Dilation Calculation

Theodore Tomalty

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1 Calculate Expression

I will use the $(+---)$ signature.

$$\mathcal{K}^\mu = (\eta, x, y, z)$$

and[1]

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (g^{\sigma\rho} \nabla_\sigma \phi \nabla_\rho \phi - m^2 \phi^2)$$

$$\begin{aligned}
D &= \int d^3x \sqrt{-g} T^{0\nu} \mathcal{K}_\nu \\
&= \int d^3x \left(\frac{\ell}{\eta}\right)^4 \left[\nabla^0 \phi \nabla^\nu \phi - \frac{1}{2} g^{0\nu} (g^{\sigma\rho} \nabla_\sigma \phi \nabla_\rho \phi - m^2 \phi^2)\right] \mathcal{K}_\nu \\
&= \int d^3x \left(\frac{\ell}{\eta}\right)^4 \left[\nabla^0 \phi \nabla_\nu \phi \mathcal{K}^\nu - \frac{1}{2} \mathcal{K}^0 (g^{\sigma\rho} \nabla_\sigma \phi \nabla_\rho \phi - m^2 \phi^2)\right] \\
&= \int d^3x \left(\frac{\ell}{\eta}\right)^4 \left[\frac{\eta^2}{\ell^2} \partial_0 \phi \left(\eta \partial_0 \phi + \sum_{j=1}^{3} x_j \partial_j \phi\right) - \frac{1}{2}(\eta) \left(\frac{\eta^2}{\ell^2} \left(\partial_0 \phi^2 - (\nabla \phi)^2\right) - m^2 \phi^2\right)\right] \\
&= \int d^3x \left(\frac{\ell}{\eta}\right)^4 \left[\frac{\eta}{2} \left(\frac{\eta^2}{\ell^2} (\partial_0 \phi^2 + (\nabla \phi)^2) + m^2 \phi^2\right) + \frac{\eta^2}{\ell^2} \partial_0 \phi \sum_{j=1}^{3} x_j \partial_j \phi\right]
\end{aligned} \tag{1}$$

Now we write the field in terms of creation and annihilation operators:

$$\begin{aligned}
\phi(x) &= \int d^3k [a_{\mathbf{k}} u_k(x) + a^\dagger_{\mathbf{k}} u^*_k(x)] \\
u_k(x) &= e^{i\mathbf{k}\cdot\mathbf{x}} f_k(\eta), \; k = \sqrt{\mathbf{k}^2} \\
g_k(\eta) &\equiv \frac{\partial}{\partial \eta} f_k(\eta), \; h_k(\eta) \equiv \frac{\partial}{\partial k} f_k(\eta)
\end{aligned}$$

And we can write out some consequences.

$$\begin{aligned}
\partial_0 \phi &= \int d^3k [a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} g_k(\eta) + a^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} g^*_k(\eta)] \\
\partial_j \phi &= \int d^3k [a_{\mathbf{k}} u_k(x) - a^\dagger_{\mathbf{k}} u^*_k(x)](ik_j)
\end{aligned}$$

[1] The expression for energy-momentum is on page 12 of Fewster notes, using the same signature.

$$\begin{aligned}
f_k(\eta) &= \beta(\eta) H_\nu^{(2)}(k\eta), \;\; \beta(\eta) \propto \eta^{3/2} \\
h_k(\eta) &= \beta(\eta) H_\nu^{\prime(2)}(k\eta)\eta \\
g_k(\eta) &= \beta(\eta) H_\nu^{\prime(2)}(k\eta)k + \left(\frac{d}{d\eta}\beta(\eta)\right) H_\nu^{(2)}(k\eta)k \\
&= \beta(\eta) H_\nu^{\prime(2)}(k\eta)k + \frac{3}{2\eta}\beta(\eta) H_\nu^{(2)}(k\eta)k \\
h_k(\eta) &= g_k(\eta)\left(\frac{\eta}{k}\right) - \frac{3}{2k} f_k(\eta)
\end{aligned} \tag{2}$$

We now evaluate each terms in (1) independently.

$$\begin{aligned}
\int d^3x\phi^2 &= \int d^3x \left[\int d^3k[a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} f_k(\eta) + a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} f_k^*(\eta)]\right]^2 \\
&= \int d^3x \left[\int d^3k[a_{\mathbf{k}} f_k(\eta) + a_{-\mathbf{k}}^\dagger f_k^*(\eta)] e^{i\mathbf{k}\cdot\mathbf{x}}\right]^2 \\
&= \int d^3k \; d^3k' [a_{\mathbf{k}} f_k(\eta) + a_{-\mathbf{k}}^\dagger f_k^*(\eta)][a_{\mathbf{k}'} f_{k'}(\eta) + a_{-\mathbf{k}'}^\dagger f_{k'}^*(\eta)] \int e^{i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')} d^3x \\
&= \int (2\pi)^3 d^3k [a_{\mathbf{k}} f_k(\eta) + a_{-\mathbf{k}}^\dagger f_k^*(\eta)][a_{-\mathbf{k}} f_k(\eta) + a_{\mathbf{k}}^\dagger f_k^*(\eta)]
\end{aligned}$$

In this derivation we have used the fact that k is invariant under the change in order of integration $\mathbf{k} \to -\mathbf{k}$, and in the last line we used $\int e^{i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')} d^3x = (2\pi)^3 \delta(\mathbf{k}+\mathbf{k}')$. The exact same derivation gives.

$$\int d^3x (\partial_0\phi)^2 = \int (2\pi)^3 d^3k [a_{\mathbf{k}} g_k(\eta) + a_{-\mathbf{k}}^\dagger g_k^*(\eta)][a_{-\mathbf{k}} g_k(\eta) + a_{\mathbf{k}}^\dagger g_k^*(\eta)]$$

We continue in a similar way:

$$\begin{aligned}
\int d^3x (\nabla\phi)^2 &= \int d^3x \sum_{j=1}^3 \left[\int d^3k[a_{\mathbf{k}} u_k(x) - a_{\mathbf{k}}^\dagger u_k^*(x)](ik_j)\right]^2 \\
&= \sum_{j=1}^3 \int d^3k \; d^3k' [a_{\mathbf{k}} f_k(\eta) + a_{-\mathbf{k}}^\dagger f_k^*(\eta)](ik_j)[a_{\mathbf{k}'} f_{k'}(\eta) + a_{-\mathbf{k}'}^\dagger f_{k'}^*(\eta)](ik'_j) \int e^{i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')} d^3x \\
&= \int d^3k (2\pi)^3 k^2 [a_{\mathbf{k}} f_k(\eta) + a_{-\mathbf{k}}^\dagger f_k^*(\eta)][a_{-\mathbf{k}} f_k(k) + a_{\mathbf{k}}^\dagger f_k^*(k)]
\end{aligned}$$

Note that in the second line the $-$ signs become positive because there is a factor of k_j that becomes negative under the change of integration $\mathbf{k} \to -\mathbf{k}$ for the second term in the brackets. The next calculation is complicated slightly by the appearance of the spacial x_j terms.

$$\int d^3x \partial_0\phi \sum_{j=1}^3 \partial_j\phi x_j = \sum_{j=1}^3 \int d^3k \; d^3k' [a_{\mathbf{k}} g_k(\eta) + a_{-\mathbf{k}}^\dagger g_k^*(\eta)][a_{\mathbf{k}'} f_{k'}(\eta) + a_{-\mathbf{k}'}^\dagger f_{k'}^*(\eta)](ik'_j) \int e^{i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')} x_j d^3x$$

We need two identities:

$$\begin{aligned}
\int e^{i\mathbf{x}\cdot\mathbf{y}} x_j d^3x &= -i(2\pi)^3 \frac{d}{dy_j} \delta^{(3)}(\mathbf{y}) \\
\frac{d}{dk_j} f_k(\eta) &= h_k(\eta) \frac{dk}{dk_j} \\
&= h_k(\eta) \frac{k_j}{k}
\end{aligned}$$

Using these the above expression becomes, after moving the derivative off the delta function (remember to introduce a factor of -1 when this happens):

$$\int d^3x\partial_0\phi\sum_{j=1}^{3}\partial_j\phi x_j = -\sum_{j=1}^{3}\int d^3k\ d^3k'[a_{\mathbf{k}}g_k(\eta)+a^\dagger_{-\mathbf{k}}g^*_k(\eta)]\frac{d}{dk'_j}\left([a_{\mathbf{k}'}f_{k'}(\eta)+a^\dagger_{-\mathbf{k}'}f^*_{k'}(\eta)](k'_j)\right)_{\mathbf{k}'=-\mathbf{k}}(2\pi)^3$$

$$\begin{aligned}\sum_{j=1}^{3}\frac{d}{dk_j}\left([a_{\mathbf{k}}f_k(\eta)+a^\dagger_{-\mathbf{k}}f^*_k(\eta)](k_j)\right) &= \sum_{j=1}^{3}\left(\partial_j a_{\mathbf{k}}f_k(\eta)-\partial_j a^\dagger_{-\mathbf{k}}f^*_k(\eta)+a_{\mathbf{k}}h_k(\eta)\frac{k_j}{k}+\partial_j a^\dagger_{-\mathbf{k}}h^*_k(\eta)\frac{k_j}{k}\right)k_j\\ &\quad+3[a_{\mathbf{k}}f_k(\eta)+a^\dagger_{-\mathbf{k}}f^*_k(\eta)]\\ &=\sum_{j=1}^{3}\left(\partial_j a_{\mathbf{k}}f_k(\eta)k_j-\partial_j a^\dagger_{-\mathbf{k}}f^*_k(\eta)k_j\right)+a_{\mathbf{k}}h_k(\eta)k+a^\dagger_{-\mathbf{k}}h^*_k(\eta)k\\ &\quad+3[a_{\mathbf{k}}f_k(\eta)+a^\dagger_{-\mathbf{k}}f^*_k(\eta)]\\ &=\sum_{j=1}^{3}\left(\partial_j a_{\mathbf{k}}f_k(\eta)k_j-\partial_j a^\dagger_{-\mathbf{k}}f^*_k(\eta)k_j\right)\\ &\quad+a_{\mathbf{k}}g_k(\eta)\eta+a^\dagger_{-\mathbf{k}}g^*_k(\eta)\eta\\ &\quad+\frac{3}{2}[a_{\mathbf{k}}f_k(\eta)+a^\dagger_{-\mathbf{k}}f^*_k(\eta)]\end{aligned}$$

Where in the last line we have used the expression in (2). Finally,

$$\begin{aligned}\int d^3x\partial_0\phi\sum_{j=1}^{3}\partial_j\phi x_j = \sum_{j=1}^{3}\int d^3k(2\pi)^3[a_{\mathbf{k}}g_k(\eta)+a^\dagger_{-\mathbf{k}}g^*_k(\eta)]&\Bigg(\sum_{j=1}^{3}\left(\partial_j a_{-\mathbf{k}}f_k(\eta)k_j-\partial_j a^\dagger_{\mathbf{k}}f^*_k(\eta)k_j\right)\\ &-(a_{-\mathbf{k}}g_k(\eta)+a^\dagger_{\mathbf{k}}g^*_k(\eta))\eta-\frac{3}{2}(a_{-\mathbf{k}}f_k(\eta)+a^\dagger_{\mathbf{k}}f^*_k(\eta))\Bigg)\end{aligned}$$

And all that is left to do is put all the terms together[2].

$$\begin{aligned}D=\left(\frac{\ell}{\eta}\right)^4\int d^3k(2\pi)^3\Bigg[&\frac{\eta}{2}\left(\frac{\eta^2}{\ell^2}k^2+m^2\right)[a_{\mathbf{k}}f_k(\eta)+a^\dagger_{-\mathbf{k}}f^*_k(\eta)][a_{-\mathbf{k}}f_k(\eta)+a^\dagger_{\mathbf{k}}f^*_k(\eta)]\\ &+\frac{\eta^2}{\ell^2}[a_{\mathbf{k}}g_k(\eta)+a^\dagger_{-\mathbf{k}}g^*_k(\eta)]\Bigg(\sum_{j=1}^{3}\left(\partial_j a_{-\mathbf{k}}f_k(\eta)k_j-\partial_j a^\dagger_{\mathbf{k}}f^*_k(\eta)k_j\right)\\ &-\frac{1}{2}(a_{-\mathbf{k}}g_k(\eta)+a^\dagger_{\mathbf{k}}g^*_k(\eta))\eta-\frac{3}{2}(a_{-\mathbf{k}}f_k(\eta)+a^\dagger_{\mathbf{k}}f^*_k(\eta))\Bigg)\Bigg]\end{aligned}$$

Now Expand, remembering to combine the $a^\dagger_{\mathbf{k}}a_{\mathbf{k}}$ and $a_{\mathbf{k}}a^\dagger_{\mathbf{k}}$ terms.

$$\begin{aligned}D=\int d^3k(2\pi)^3\left(\frac{\ell}{\eta}\right)^4\Bigg[&a^\dagger_{\mathbf{k}}a_{\mathbf{k}}A^{(1)}_k(\eta)+a_{\mathbf{k}}a_{-\mathbf{k}}A^{(2)}_k(\eta)+a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}A^{*(2)}_k(\eta)\\ &+\frac{\eta^2}{\ell^2}\Bigg(g_k(\eta)f_k(\eta)a_{\mathbf{k}}\sum_{j=1}^{3}\partial_j a_{-\mathbf{k}}k_j-g_k(\eta)f^*_k(\eta)a_{\mathbf{k}}\sum_{j=1}^{3}\partial_j a^\dagger_{\mathbf{k}}k_j\\ &-g^*_k(\eta)f_k(\eta)a^\dagger_{\mathbf{k}}\sum_{j=1}^{3}\partial_j a_{\mathbf{k}}k_j+g^*_k(\eta)f^*_k(\eta)a^\dagger_{\mathbf{k}}\sum_{j=1}^{3}\partial_j a^\dagger_{-\mathbf{k}}k_j\Bigg)\Bigg]\end{aligned}\tag{3}$$

[2]Note that the expression for $\int d^3x(\partial_0\phi)^2$ combines nicely with that of $\int d^3x\partial_0\phi\sum_{j=1}^{3}\partial_j\phi x_j$

And the coefficients for the non-derivative operators are defined as follows:

$$
\begin{aligned}
A_k^{(1)}(\eta) &= \eta|f_k(\eta)|^2\left(\frac{\eta^2}{\ell^2}k^2+m^2\right)-\frac{\eta^2}{\ell^2}\left(|g_k(\eta)|^2\eta+\frac{3}{2}\left[g_k^*(\eta)f_k(\eta)+g_k(\eta)f_k^*(\eta)\right]\right)\\
A_k^{(2)}(\eta) &= \frac{\eta}{2}f_k(\eta)^2\left(\frac{\eta^2}{\ell^2}k^2+m^2\right)-\frac{\eta^2}{\ell^2}\left(\frac{1}{2}g_k(\eta)^2\eta+\frac{3}{2}g_k(\eta)f_k(\eta)\right)
\end{aligned}
\tag{4}
$$

2 Simplify Expression

Now it is time to try and combine all of these pesky terms. For one thing, we will make much use of the identity $f_k g_k^* = f_k^* g_k + \frac{i}{(2\pi)^3}\left(\frac{\eta}{\ell}\right)$. First we can select out particular terms in the expression for D.

$$
\begin{aligned}
D &= D' + \int d^3k(2\pi)^3\left(\frac{\ell}{\eta}\right)^2\left[-g_k f_k^* a_{\mathbf{k}}\sum_j \partial_j a_{\mathbf{k}}^\dagger k_j - g_k^* f_k a_{\mathbf{k}}^\dagger \sum_j \partial_j a_{\mathbf{k}} k_j\right]\\
&= D' + \int d^3k(2\pi)^3\left(\frac{\ell}{\eta}\right)^2\left[-g_k^* f_k a_{\mathbf{k}}\sum_j \partial_j a_{\mathbf{k}}^\dagger k_j - g_k f_k^* a_{\mathbf{k}}^\dagger \sum_j \partial_j a_{\mathbf{k}} k_j\right]\\
&- i\int d^3k\sum_{j=1}^{3} k_j a_{\mathbf{k}}^\dagger \overleftrightarrow{\partial_j} a_{\mathbf{k}}
\end{aligned}
$$

This last term looks like it should come out in the end so we keep the expression like this (even though it may seem more complicated than it was a second ago). Now we can take some other terms in D' that look like they might help cancel these terms.

$$
\begin{aligned}
D' &= D'' + \int d^3k(2\pi)^3\left(\frac{\ell}{\eta}\right)^4\left[a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\left(-\left(\frac{\eta}{\ell}\right)^2 g_k g_k^*\eta\right)\right]\\
&= D'' - (2\pi)^3\left(\frac{\ell}{\eta}\right)^2\int d^3k\left[a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\left(\frac{1}{2}(kh_k+\frac{3}{2}f_k)g_k^* + \frac{1}{2}(kh_k^*+\frac{3}{2}f_k^*)g_k\right)\right]
\end{aligned}
$$

Now note that there are also terms like $a_{\mathbf{k}}^\dagger a_{\mathbf{k}}(f_k g_k^* + f_k^* g_k)$ in D'' so we can rewrite $D'' \to \tilde{D}''$ by absorbing those terms (i.e. change the coefficients in $\frac{3}{2} \to \frac{9}{4}$ in $A_k^{(1)}(\eta)$. We can also see that $kh_k = \sum_j \partial_j f_k k_j$ and we can now do a 3-way integration by parts on these terms.

$$
\begin{aligned}
D' &= \tilde{D}'' - (2\pi)^3\left(\frac{\ell}{\eta}\right)^2\int d^3k a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\frac{1}{2}\left[kh_k g_k^* + kh_k^* g_k\right]\\
&= \tilde{D}'' + \left[\int \partial_j(g_k k_j)\right]_{\text{terms}} + (2\pi)^3\left(\frac{\ell}{\eta}\right)^2\int d^3k\frac{1}{2}\left[(f_k g_k^* + f_k^* g_k)\sum_j(a_{\mathbf{k}}^\dagger\partial_j a_{\mathbf{k}} + \partial_j a_{\mathbf{k}}^\dagger a_{\mathbf{k}})k_j\right]\\
&= \tilde{D}'' + \left[\int \partial_j(g_k k_j)\right]_{\text{terms}} + (2\pi)^3\left(\frac{\ell}{\eta}\right)^2\int d^3k\left[(f_k^* g_k\sum_j a_{\mathbf{k}}^\dagger\partial_j a_{\mathbf{k}} k_j + f_k g_k^*\sum_j \partial_j a_{\mathbf{k}}^\dagger a_{\mathbf{k}} k_j\right]\\
&+ \frac{1}{2}i\int d^3k\sum_{j=1}^{3} k_j a_{\mathbf{k}}^\dagger \overleftrightarrow{\partial_j} a_{\mathbf{k}}\\
D &= \tilde{D}'' + \left[\int \partial_j(g_k k_j)\right]_{\text{terms}} - \frac{i}{2}\int d^3k\sum_{j=1}^{3} k_j a_{\mathbf{k}}^\dagger \overleftrightarrow{\partial_j} a_{\mathbf{k}}
\end{aligned}
$$

As was suspected, the hanging terms in the first simplification vanished, and our expression is greatly simplified. Note that so far we have not touched any of the terms in D that involve $\mathbf{k}$ and $-\mathbf{k}$, and for now we can sweep all of those into the term, $D^{(3)}$, until later. Let us now examine the remaining terms involving $a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$.

$$\left[\int \partial_j (g_k k_j)\right]_{\text{terms}} = (2\pi)^3 \left(\frac{\ell}{\eta}\right)^2 \int d^3k a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \frac{1}{2}\left[f_k \sum_j \partial(g^*_k k_j) + f^*_k \sum_j \partial(g_k k_j)\right]$$

$$D = D^{(3)} + \int d^3x a^\dagger_{\mathbf{k}} a_{\mathbf{k}} B_k(\eta) - \frac{i}{2}\int d^3k \sum_{j=1}^{3} k_j a^\dagger_{\mathbf{k}} \overleftrightarrow{\partial}_j a_{\mathbf{k}}$$

$$B_k(\eta) = (2\pi)^3 \left(\frac{\ell}{\eta}\right)^4 \left[\eta f_k f^*_k \left(\left(\frac{\eta}{\ell}\right)^2 k^2 + m^2\right) - \frac{3}{4}\left(\frac{\eta}{\ell}\right)^2 (g^*_k f_k + g_k f^*_k) + \frac{1}{2}\left(\frac{\eta}{\ell}\right)^2 \left(f_k \sum_j k_j \partial g^*_k + f^*_k \sum_j k_j \partial g_k\right)\right]$$

Note the factor of $\frac{3}{4}$ rather than than $\frac{9}{4}$, from the discussion of $\tilde{D}''$, because we have combined the coefficient in $A^{(1)}_k$ with the $\frac{1}{2} f_k g^*_k \sum_j \partial_j k_j = \frac{6}{4} f_k g^*_k$ term.

We now, however, have a problem because we are left with derivatives of g_k which we do not have any nice identities for. But we are saved by the fact that $f_k = \alpha(\eta)\chi_k$ and χ_k satisfies a second order equation of motion.

$$\chi_k = \frac{1}{2}\sqrt{\pi\eta} H^{(2)}_\nu (k\eta)$$

$$g_k(\eta) = \frac{1}{\eta}\alpha(\eta)\chi_k + \alpha(\eta)\chi'_k$$

So in order to calculate $\partial_j g_k$ we must simply evaluate $\partial_j \chi_k$ and $\partial_j \chi'_k$ in terms of Hankel functions and derivatives of Hankel functions and then write the answer in η derivatives of $\chi_k(\eta)$. One can check that in the end we get

$$\partial_j g_k = \alpha(\eta)\frac{k_j}{k^2}\left[\frac{-1}{2\eta}\chi_k + \frac{3}{2}\chi'_k + \eta\chi''_k\right]$$

And we continue:

$$\sum_j k_j \partial_j g_k = \alpha(\eta)\left[\frac{-1}{2\eta}\chi_k + \frac{3}{2}\chi'_k + \eta\chi''_k\right]$$

$$\chi''_k(\eta) = -\left[k^2 + \left(\frac{\ell}{\eta}\right)^2 (m^2 - \xi(n)R)\right]\chi_k(\eta) \quad = -\left[k^2 + \left(\frac{\ell}{\eta}\right)^2 (m^2 - \frac{2}{\ell^2})\right]\chi_k(\eta)$$

Now, since we can get rid of the second order time derivative of χ_k we can move back to using combinations of g_k and f_k.

$$\sum_j k_j \partial_j g_k = g_k + \frac{1}{2}\left(g_k - \frac{1}{\eta} f_k\right) - \eta\left(k^2 + \left(\frac{\ell}{\eta}\right)^2 m^2\right) f_k + \frac{2}{\eta} f_k - \frac{3}{2\eta} f_k$$

$$= \frac{3}{2} g_k - \eta\left(k^2 + \left(\frac{\ell}{\eta}\right)^2 m^2\right) f_k$$

$$\frac{1}{2}\left(\frac{\eta}{\ell}\right)^2 \left(f_k \sum_j k_j \partial g^*_k + f^*_k \sum_j k_j \partial g_k\right) = \left(\frac{\eta}{\ell}\right)^2 \left[\frac{3}{4}(f_k g^*_k + f^*_k g_k) - \eta\left(k^2 + \left(\frac{\ell}{\eta}\right)^2 m^2\right)\right]$$

Suddenly all the terms in $B_k(\eta)$ cancel with each each other and we have $B_k(\eta) = 0$. Remembering that we defined $D^{(3)}$ as all the contributions to D that involve $\pm\mathbf{k}$, we now have:

$$D = D^{(3)} - \frac{i}{2}\int d^3k \sum_{j=1}^{3} k_j a^\dagger_{\mathbf{k}} \overleftrightarrow{\partial}_j a_{\mathbf{k}}$$

When dealing with this $D^{(3)}$ we can no longer move the complex conjugates around because there are no $f_k g_k^*$ terms to work with, but we can still integrate by parts like before and hope for the best. Recall that D is Hermitian so we can focus our attention on the terms of $D^{(3)}$ that do not have any conjugates, Hermitian or otherwise. We consider one $a_{\mathbf{k}} a_{-\mathbf{k}} A_k^{(2)}$ term and one $a_{\mathbf{k}}\partial_j a_{-\mathbf{k}}$ term.

$$\begin{aligned} D^{(3)} &= D^{(4)} - (2\pi)^3 \left(\frac{\ell}{\eta}\right)^2 \int d^3k a_{\mathbf{k}} a_{-\mathbf{k}} \frac{1}{2} g_k^2 \eta \\ &= D^{(4)} - (2\pi)^3 \left(\frac{\ell}{\eta}\right)^2 \int d^3k a_{\mathbf{k}} a_{-\mathbf{k}} \frac{1}{2} \left(k h_k + \frac{3}{2} f_k\right) g_k \eta \end{aligned}$$

Once more, we absorb the above $f_k g_k$ term into $D^{(4)} \to \tilde{D}^{(4)}$ by doing $\frac{3}{2} \to \frac{9}{4}$ in $A_k^{(2)}(\eta)$. So

$$\begin{aligned} D^{(3)} &= \tilde{D}^{(4)} - (2\pi)^3 \left(\frac{\ell}{\eta}\right)^2 \int d^3k a_{\mathbf{k}} a_{-\mathbf{k}} \frac{1}{2} \sum_j \partial_j (f_k) k_j g_k \eta \\ &= \tilde{D}^{(4)} - (2\pi)^3 \left(\frac{\ell}{\eta}\right)^2 \frac{1}{2} \int d^3k \left(a_{\mathbf{k}} a_{-\mathbf{k}} \left[3 f_k g_k + f_k \sum_j k_j \partial_j g_k \right] + f_k g_k \sum_j k_j (-a_{\mathbf{k}} \partial_j a_{-\mathbf{k}} + \partial_j a_{\mathbf{k}} a_{-\mathbf{k}}) \right) \end{aligned}$$

and note that in the last line a negative sign appears because $\partial_j a_{-\mathbf{k}}$ is shorthand for

$$\left.\frac{\partial}{\partial k'_j} a_{\mathbf{k}'}\right|_{\mathbf{k}'=-\mathbf{k}}$$

Since $a_{\mathbf{k}}$ and $a_{-\mathbf{k}}$ commute, we can invert the integration $k_j \partial_j a_{\mathbf{k}} a_{-\mathbf{k}} \to -k_j a_{\mathbf{k}} \partial_j a_{-\mathbf{k}}$ and viola! The terms in $D^{(3)}$ above that involve derivatives of $a_{\mathbf{k}}$ exactly cancel with the derivative terms in $\tilde{D}^{(3)}$. We are left with

$$D^{(3)} = (2\pi)^3 \left(\frac{\ell}{\eta}\right)^2 \int d^3k a_{\mathbf{k}} a_{-\mathbf{k}} \left[\frac{\eta}{2} f_k^2 \left(k^2 + \left(\frac{\ell}{\eta}\right)^2 m^2\right) - \frac{3}{4} g_k f_k + \frac{1}{2} f_k \sum_j k_j \partial_j g_k \right]$$

But from our earlier analysis we know that

$$\frac{1}{2} f_k \sum_j k_j \partial_j g_k = \frac{3}{4} g_k f_k - \frac{\eta}{2} \left(k^2 + \left(\frac{\ell}{\eta}\right)^2 m^2\right) f_k^2$$

So all terms in $D^{(3)}$ cancel with each other and we are left with $D^{(3)} = 0$. Or, in other words, we finally have

$$D = \frac{-i}{2} \int d^3k \sum_{j=1}^{3} k_j a^\dagger_{\mathbf{p}} \overleftrightarrow{\partial}_j a_{\mathbf{p}}, \tag{5}$$

3 Trouble with the Zero Mode

In the general case of a Robertson-Walker spacetime, the fundamental modes are given by the second order differential equation:

$$
\begin{aligned}
u_{\mathbf{k}}(\mathbf{x},\eta) &\propto e^{i\mathbf{k}\cdot\mathbf{x}}C(\eta)^{(2-n)/4}\chi_{\mathbf{k}}(\eta) \\
\frac{d^2\chi_{\mathbf{k}}}{d\eta^2} &+ \{k^2 + C(\eta)[m^2 + (\xi - \xi(n))R(\eta)]\}\chi_{\mathbf{k}} = 0 \\
\xi(n) &= \frac{n-2}{4(n-1)}
\end{aligned}
$$

For the case of de Sitter space, with $k = 0$, $C(\eta) = \frac{\eta^2}{\ell^2}$, and $R(\eta) = \frac{12}{\ell^2}$ we get:

$$
\begin{aligned}
u_0 &= \eta\chi_0(\eta) \\
\frac{d^2\chi_0}{d\eta^2} &+ \frac{\eta^2}{\ell^2}\left[m^2 - \frac{2}{\ell^2}\right]\chi_0 = 0
\end{aligned}
$$

which has solutions

$$
u_\pm(\mathbf{x},\eta) = \eta^{\omega_\pm}, \text{ with } \omega_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \ell^2 m^2} \tag{6}
$$

The natural thing to do now is to add these modes to the field mode expansion, and make the assumption that $a_{\mathbf{k}}$ vanishes around $\mathbf{k} = 0$. And then we are able add these extra terms to the expression for the dilation charge.

$$
\phi(x) = \int d^3k[a_{\mathbf{k}} f_k(\eta) + a^\dagger_{-\mathbf{k}} f^*_k(\eta)]e^{i\mathbf{k}\cdot\mathbf{x}} + a_+u_+(\eta) + a_-u_-(\eta)
$$

$$
\begin{aligned}
\int d^3x\phi^2 \mathrel{+}= &\left\{a_+u_+(\eta) + a_-u_-(\eta), \int d^3k[a_{\mathbf{k}} f_k(\eta) + a^\dagger_{-\mathbf{k}} f^*_k(\eta)]\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\right\} \\
&+ \int d^3x(a_+u_+(\eta) + a_-u_-(\eta))^2
\end{aligned}
$$

The first term vanishes because $\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}$ reduces to a Dirac delta function centred at $k = 0$, and the second term is constant in the integrand, so

$$
\int d^3x\phi^2 \mathrel{+}= V(a_+u_+(\eta) + a_-u_-(\eta))^2
$$

Following the same logic we also get,

$$
\int d^3x(\partial_0\phi)^2 \mathrel{+}= \frac{V}{\eta^2}(\omega_+a_+u_+(\eta) + \omega_-a_-u_-(\eta))^2
$$

The $u_\pm$ modes are constant in space so they do not contribute to the $\int d^3x(\nabla\phi)^2$ term, but there is one more term in D that we must consider:

$$
\int d^3x\partial_0\phi\sum_{j=1}^{3}\partial_j\phi x_j \mathrel{+}= \frac{1}{\eta}(\omega_+a_+u_+(\eta) + \omega_-a_-u_-(\eta))\sum_{j=1}^{3}\int d^3k[a_{\mathbf{k}} f_k(\eta) + a^\dagger_{-\mathbf{k}} f^*_k(\eta)](ik_j)\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}x_j
$$

Here the spacial integral converges to a derivative of the Dirac delta. When it is applied to the k-space integrand, the product rule expansion has both terms that vanish at $k = 0$. So,

$$
\int d^3x\partial_0\phi\sum_{j=1}^{3}\partial_j\phi x_j \mathrel{+}= 0
$$

Now with all of these terms, we can calculate the zero mode contribution to D using (1)

$$
\begin{aligned}
D += & \left(\frac{\ell}{\eta}\right)^4 \left(\frac{\eta}{2}\right) \left(\frac{\eta^2}{\ell^2}\frac{V}{\eta^2}(\omega_+ a_+ u_+(\eta) + \omega_- a_- u_-(\eta))^2 + m^2 V (a_+ u_+(\eta) + a_- u_-(\eta))^2\right) \\
= & \frac{V\ell^2}{2\eta^3}\left(a_+^2 u_+^2(\eta)(\omega_+^2 + \ell^2 m^2) + a_-^2 u_-^2(\eta)(\omega_-^2 + \ell^2 m^2) + (a_+ a_- + a_- a_+)(\omega_+\omega_- + \ell^2 m^2) u_+ u_-\right)
\end{aligned}
$$

It can be shown that the following identities hold:

$$
\begin{aligned}
\omega_+\omega_- &= \ell^2 m^2 \\
u_+ u_- &= \eta^3 \\
(\omega_\pm - 1)(\omega_\pm - 2) &= 2 - \ell^2 m^2 \\
\omega_\pm^2 + \ell^2 m^2 &= 3\omega_\pm
\end{aligned}
$$

where the fourth line follows from the third. We can then simplify the zero-mode contribution to D to

$$
\begin{aligned}
D += & \frac{V\ell^2}{2\eta^3}\left(a_+^2 u_+^2(\eta) 3\omega_+ + a_-^2 u_-^2(\eta) 3\omega_- + 2\ell^2 m^2 (a_+ a_- + a_- a_+)\eta^3\right) \\
= & V\frac{3}{2}\ell^2\left(a_+^2 \eta^{2\nu}\omega_+ + a_-^2 \eta^{-2\nu}\omega_-\right) + V\ell^4 m^2 (a_+ a_- + a_- a_+)
\end{aligned}
$$

Note that we have used $\nu = \sqrt{\frac{9}{4} - \ell^2 m^2}$, which is the index of the Hankel functions in section 1, and also that this analysis is valid whether or not ν is real or imaginary[3].

4 Zero Mode Analysis

As was mentioned in the previous footnote, when ν is real $a_\pm$ are Hermition, which is not something that we want for creation and annihilation operators. Instead we will go about finding $u_0(\eta)$ in a more formal way.

We have the solutions to the differential equation at $k = 0$ and so we write the general solution:

$$u_0(\eta) = A\eta^{\omega_-} + B\eta^{\omega_+}$$

Now we us the inner product defined by

$$(\phi, \pi) = -i\int d^3x\sqrt{-g}\,\phi\overleftrightarrow{\partial_0}\pi^* n^0$$

And set the condition

$$
\begin{aligned}
(u_0, u_0) &= 1 \\
(u_0, u_0^*) &= 0
\end{aligned}
$$

So

$$
\begin{aligned}
1 &= -i\int d^3x\sqrt{-g}\,u_0\overleftrightarrow{\partial_0}u_0^*\left(\frac{\eta^2}{\ell^2}\right) \\
&= -i\int d^3x\left(\frac{\ell^2}{\eta^2}\right)\left[(A\eta^{\omega_+} + B\eta^{\omega_-})(\omega_+ A^*\eta^{\omega_+} + \omega_- B^*\eta^{\omega_-}) - (A^*\eta^{\omega_+} + B^*\eta^{\omega_-})(\omega_+ A\eta^{\omega_+} + \omega_- B\eta^{\omega_-})\right]\frac{1}{\eta} \\
&= -i\int d^3x\left(\frac{\ell^2}{\eta^2}\right)\left[\omega_- AB^* + \omega_+ A^*B - \omega_+ AB^* - \omega_- A^*B\right]\eta^2 \\
&= -i\int d^3x\left(\frac{\ell^2}{\eta^2}\right)\left[(\omega_- + \omega_+)(AB^* - BA^*)\right]\eta^2
\end{aligned}
$$

[3] If ν is real then $u_\pm(\eta)$ are both real-valued functions so $a_\pm$ must be Hermitian. Otherwise we will have $u_+(\eta) = u_-^*(\eta)$ and so $a_+ = a_-^\dagger$

And we are left with the restriction:

$$(AB^* - BA^*) = (2\nu V i\ell^2)^{-1} \tag{7}$$

Now we simplify the problem by removing an overall phase (set A real) and specifying that $|A| = |B|$. We can then write the general solution as

$$u_0(\eta) = A(\eta^{\omega_+} + e^{i\gamma}\eta^{\omega_-})$$

From this, equation 7 requires that

$$2A^2 \sin(\gamma) = (2\nu V \ell^2)^{-1}$$

You can check that the condition $(u_0, u_0^*) = 0$ does not give any more requirements for the parameters, so we are free to choose an arbitrary γ such that $\sin(\gamma) \neq 0$. The obvious choice is $\gamma = \frac{\pi}{4}$ since any other quantity would just add to the real part and skew the real-imaginary weights. Therefore

$$\begin{aligned} u_0 &= A(\eta^{\omega_+} + i\eta^{\omega_-}) \\ A &= \frac{1}{2\ell}\sqrt{\frac{1}{\nu V}} \end{aligned} \tag{8}$$

4.1 Checking Canonical Commutation Relation

We can now define the field more precisely by:

$$\phi(x) = \int_{\Sigma\backslash B(\epsilon)} d^3k \left[a_{\mathbf{k}} f_k + a^\dagger_{-\mathbf{k}} f_k^*\right] e^{i\mathbf{k}\cdot\mathbf{x}} + a_0 u_0(\eta) + a_0^\dagger u_0^*(\eta) \tag{9}$$

Since the zero mode operators do not commute with those in the integral we have

$$\begin{aligned}[][\phi(x), \pi(x')] = \left(\frac{\ell^2}{\eta^2}\right)\Bigg[\int_{\Sigma\backslash B(\epsilon)}\int_{\Sigma\backslash B(\epsilon)} d^3k d^3k' \left[[a_{\mathbf{k}}, a^\dagger_{-\mathbf{k}'}] f_k g^*_{k'} + [a^\dagger_{-\mathbf{k}}, a_{\mathbf{k}'}] f_k^* g_{k'}\right] e^{i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'\cdot\mathbf{x}'} \\ + \left[[a_0, a_0^\dagger] u_0 \partial_0 u_0^* + [a_0^\dagger, a_0] u_0^* \partial_0 u_0\right]\Bigg]\end{aligned}$$

But we still have $[a_{\mathbf{k}}, a^\dagger_{-\mathbf{k}'}] = \delta(k + k')$ and $[a_0, a_0^\dagger] = 1$, so the above equation reduces to

$$[\phi(x), \pi(x')] = \left(\frac{\ell^2}{\eta^2}\right)\left[\int_{\Sigma\backslash B(\epsilon)} d^3k \left[f_k g_k^* - f_k^* g_k\right] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} + u_0 \overleftrightarrow{\partial_0} u_0^*\right]$$

Now $[f_k g_k^* - f_k^* g_k]$ will be constant in the entire domain of integration. If we were integrating over all of space this would integrate to a delta function, but in the limit where ϵ is small we have $e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \approx 1$ so we can remove from the delta function a small correction. We can also evaluate the last therm by using the definition of our zero-mode solution from (8) to get,

$$[\phi(x), \pi(x')] = \frac{i}{(2\pi)^3}\left(\frac{\eta^2}{\ell^2}\right)\left[(2\pi)^3\delta^{(3)}(x - x') - \frac{4}{3}\pi\epsilon^3\right] + 4iA^2\eta^2\nu$$

Therefore in order for the canonical commutation relations to be satisfied, the two extra terms must cancel and we get an equation relating the bounds of integration in real and phase space:

$$V = \frac{6\pi^2}{\epsilon^3} \tag{10}$$

4.2 Flux Through the Boundary

In our previous analysis we found that the zero-mode terms add a time-dependent component to the boost charge. Since a charge is usually conserved, the time dependence must come from some flux through the boundaries of the system. Remember we used a finite volume V in our expressions.

In the previous subsection we have changed the definition of the zero mode, and thus we will get a slightly different expression for the Dilation charge correction (although the details are the same as at the beginning of the section). We use

$$D \mathrel{+}= VA^2\left(\frac{\ell}{\eta}\right)^4\left(\frac{\eta}{2}\right)\Bigg[\Big((\omega_+\eta^{\omega_+}+i\omega_-\eta^{\omega_-})a_0+(\omega_+\eta^{\omega_+}-i\omega_-\eta^{\omega_-})a_0^\dagger\Big)^2 + \ell^2m^2\Big((\eta^{\omega_+}+i\eta^{\omega_-})a_0+(\eta^{\omega_+}-i\eta^{\omega_-})a_0^\dagger\Big)^2\Bigg] \tag{11}$$

From the discussion in the beginning we had a conserved current such that

$$\int d^4x\sqrt{-g}\nabla_\mu j^\mu = 0$$

Which we then wrote as $D_{\eta=0}-D_{\eta=\eta'}=0$ when we should have more precisely said:

$$0 = D_{\eta=0}-D_{\eta=\eta'}+\int_0^{\eta'}\int_{\partial B(R)}\rho j^\mu\cdot n_\mu d^2x d\eta$$

Since we are integrating over the surface of a hypercylinder, the normal four-vector, n^μ points in the radial direction:

$$n^\mu = \frac{\eta}{R\ell}(0,x^1,x^2,x^3) \tag{12}$$

where the coefficient in front ensures that $|n^\mu n_\mu| = 1$. Recall the expression for the conserved current from the beginning,

$$\begin{aligned} j_\mu &= T_{\mu\nu}\mathcal{K}^\nu \\ &= \frac{-1}{2}g_{\mu\nu}x^\nu\left(\nabla^\sigma\phi\nabla_\sigma\phi - m^2\phi^2\right)+\nabla_\mu\phi\nabla_\nu\phi x^\nu \\ j_\mu n^\mu &= \frac{1}{2}\left(\nabla^\sigma\phi\nabla_\sigma\phi - m^2\phi^2\right)\left(\frac{\ell^2}{\eta^2}\right)\left(\frac{-R\eta}{\ell}\right)-\left(\frac{\eta}{R\ell}\right)\nabla_i\phi x^i\nabla_\nu\phi x^\nu \end{aligned}$$

Let us first investigate what happens for the simple ϕ^2 term. Consider first the continuous component of ϕ:

$$\int_{\partial B(R)} d^2x\phi(x) = \int_{\Sigma\backslash B(\epsilon)} d^3k\left(a_{\mathbf{k}}f_k + a^\dagger_{-\mathbf{k}}f_k^*\right)\int_{\partial B(R)} d^2x e^{i\mathbf{k}\cdot\mathbf{x}}$$

But now we use polar coordinates in the $\mathbf{k}$ to simplify the spacial integral

$$\begin{aligned} \int_{\partial B(R)} d^2x e^{i\mathbf{k}\cdot\mathbf{x}} &= \int d\phi d\theta e^{ikR\cos(\theta)}R^2\sin(\theta) \\ &= 2\pi R\frac{e^{ikR}-e^{ikR}}{ikR} \\ &= 4\pi R\frac{\sin(kR)}{kR} \end{aligned}$$

Note, however, that this is an oscillatory function of k and, since R is large, should average to zero when integrated in k-space. Using this argument we can say that the only terms surviving in $j_\mu n^\mu$ come from the zero-mode (whose spacial part integrates to $4\pi R$). This is particularly nice because it means we can ignore all the terms with spacial derivatives. We use

$$\int_{\partial B(R)} \phi^2 d^2x \approx 4\pi R(a_0 u_0 + a_0^\dagger u_0^*)^2$$
$$\int_{\partial B(R)} \dot\phi^2 d^2x \approx 4\pi R(a_0 \partial_0 u_0 + a_0^\dagger \partial_0 u_0^*)^2$$

$$\begin{aligned}
\text{Flux} &= \int_{\partial B(R)} \phi^2 d^2x \left(\frac{\ell}{\eta}\right)^3 j_\mu n^\mu \\
&= 4\pi R \left(\frac{\ell}{\eta}\right)^3 \frac{1}{2} \left(\frac{\ell}{\eta}\right)^2 \left(\frac{-R\eta}{\ell^3}\right) \left[\left(a_0 A(\omega_+ \eta^{\omega_+} + i\omega_- \eta^{\omega_-}) + a_0^\dagger A(\omega_+ \eta^{\omega_+} - i\omega_- \eta^{\omega_-})\right)^2 - \ell^2 m^2 (a_0 u_0 + a_0^\dagger u_0^*)^2\right]
\end{aligned}$$

From there it is only a matter of differentiation to check that

$$\frac{\partial}{\partial \eta} D(\eta) = -\text{Flux}$$

5 Generalizing to Special Conformal Symmetries

We will attempt to apply the same analysis of the Dilation Charge to the Special Conformal symmetries in De Sitter space. In fact, we will consider only the Killing vector,

$$\mathcal{K}_{s_1}^\mu = (2x^1x^0, 2(x^1)^2 + (x^0)^2 - ((x^1)^2 + (x^2)^2 + (x^3)^2), 2x^1x^2, 2x^1x^3) \tag{13}$$

because the other two symmetries are identical up to a relabelling of the spacial axes.

Now we assume that, as with the case of the dilation charge, the special conformal charge, S_1, with this Killing vector has been computed and is conserved when the zero mode is neglected. Then it is a matter of finding the (possibly time dependent) corrections to this charge.

5.1 First Order Corrections

Here we use $\phi = u_0 a_{\mathbf{k}} + u_0^* a_{-\mathbf{k}}$ to calculate the contribution of the pure zero mode to the charge S_1.

$$\begin{aligned}
S_1 &\mathrel{+}= \int d^3x \sqrt{-g} T^{0\nu} \mathcal{K}_\nu \\
&\mathrel{+}= -\int d^3x \left(\frac{\ell}{\eta}\right)^2 T_{0\nu} \mathcal{K}_{s_1}^\nu \\
T_{0\nu} \mathcal{K}_{s_1}^\nu &= \partial_0\phi \partial_0\phi \mathcal{K}_{s_1}^0 - \frac{1}{2} g_{00} \mathcal{K}_{s_1}^0 \left(g^{00} \partial_0\phi \partial_0\phi - m^2\phi^2\right) \\
&= \frac{1}{2} (\partial_0\phi)^2 \mathcal{K}_{s_1}^0 + \frac{1}{2} m^2 g_{00} \mathcal{K}_{s_1}^0 \phi^2 \\
&= x^1 \eta \left((\partial_0\phi)^2 + m^2 \left(\frac{\ell}{\eta}\right)^2 \phi^2\right)
\end{aligned}$$

where we have used $T^{\mu\nu}$ defined at the beginning of the text.

Note that the expression in the integrand is constant over space except for the x^1 factor. So integrating over a symmetric volume, like a large sphere or cube centred at the origin, gives

$$S_1 \mathrel{+}= 0$$

We can check that this result is consistent by measuring the flux of the zero mode through the boundary of a large sphere with radius R:

$$
\begin{aligned}
j_\mu n^\mu &= T_{\mu\nu}\mathcal{K}^\nu_{s_1} n^\mu \\
&= -\frac{1}{2} g_{ij}\mathcal{K}^i_{s_1} n^j \left(g^{00}(\partial_0\phi)^2 - m^2\phi^2\right) \\
&= \frac{1}{2}\left(2x^1R^2 + (\eta^2 - R^2)x^1\right)\left(\frac{\eta}{R\ell}\right)\left((\partial_0\phi)^2 - \left(\frac{\ell}{\eta}\right)^2 m^2\phi^2\right)
\end{aligned}
$$

just like before, the only spacial dependence in the above expression is the overall x^1 factor and so we get that

$$
\text{Flux} = \int_{\partial B(R)} d^2x \left(\frac{\ell}{\eta}\right)^3 j_\mu n^\mu = 0
$$

5.2 Second Order Correction

For the second order corrections we consider the cross terms of S_1 where there is mixing between the zero mode of the field and the integral of all other $k \neq 0$ modes. Let ϕ_1 be the zero mode and ϕ_2 be the standard mode expansion in k.

$$
\begin{aligned}
T^{\text{cross}}_{\mu\nu} &= \partial_\mu\phi_1\partial_\nu\phi_2 - \frac{1}{2}g_{\mu\nu}\left(g^{\sigma\rho}(\partial_\sigma\phi_1\partial_\rho\phi_1 + \partial_\sigma\phi_2\partial_\rho) - 2m^2\phi_1\phi_2\right) \\
S_1 \mathrel{+}&= -\int d^3x \left(\frac{\ell}{\eta}\right)^2 T^{\text{cross}}_{0\nu}\mathcal{K}^\nu_{S_1} \\
T^{\text{cross}}_{0\nu}\mathcal{K}^\nu_{S_1} &= \dot{\phi}_1(\partial_\nu\phi_2\mathcal{K}^\nu) + \left(\frac{1}{2}\dot{\phi}_1\dot{\phi}_2 + m^2\phi_1\phi_2\left(\frac{\ell}{\eta}\right)^2\right)\mathcal{K}^0
\end{aligned}
$$

Expanding these in phase space we see that the last term only has spacial dependence from x^1 in the Killing vector and $e^{i\mathbf{k}\cdot\mathbf{x}}$ from the expansion of ϕ_2. When this is integrated over space we get:

$$
\begin{aligned}
\int_V d^3x e^{i\mathbf{k}\cdot\mathbf{x}} x^1 &= \frac{-id}{dk_1}\int_V d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= -i(2\pi)^3\frac{d}{dk_1}\delta^{(3)}(\mathbf{k})
\end{aligned}
$$

When this is combined with the integrals over the $a_\mathbf{k}$ and $a_{-\mathbf{k}}$ from ϕ_2 this term vanishes because of sufficient suppression of the field (and momentum conjugate) near the origin.

We turn our attention to the first term and see that actually the zero element of the contraction behaves the same way as we just discussed so it vanishes as well. We then have:

$$
S_1 \mathrel{+}= -\int d^3x \left(\frac{\ell}{\eta}\right)^2 \dot{\phi}_1(\partial_i\phi_2\mathcal{K}^i)
$$

and here we have spacial dependence from ϕ_2 and from the Killing vector as follows

$$
\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}(ik_j)\mathcal{K}^j = i\left[k_1\left((-i)^2\frac{d^2}{dk_1^2} + \eta^2 - (-i)^2\frac{d^2}{dk_2^2} - (-i)^2\frac{d^2}{dk_3^2}\right) + k_2 2(-i)^2\frac{d^2}{dk_1dk_2} + k_3 2(-i)^2\frac{d^2}{dk_1dk_3}\right](2\pi)^3\delta^{(3)}(\mathbf{k})
$$

where we sum over the indices of $\mathbf{k}$ and replace each x^i in the Killing vector with $-i\frac{d}{dk_i}$. Once again we have this vanishing when integrated over the $a_\mathbf{k}$ and $a_{-\mathbf{k}}$ operators given sufficient suppression of the field (and derivatives) near the origin.

Now it is only left to check the flux through the boundary.

$$\text{Flux} = \int_{\partial B(R)} d^2x \left(\frac{\ell}{\eta}\right)^2 j_\mu n^\mu$$

$$j_\mu n^\mu = \dot{\phi}_1(2\eta x^1)\partial_i\phi_2 n^i - g_{i\nu}n^i\mathcal{K}^\nu\left(\left(\frac{\ell}{\eta}\right)^2\dot{\phi}_1\dot{\phi}_2 - m^2\phi_1\phi_2\right)$$

Looking at the first term we see that the spacial dependence gives

$$\int_{\partial B(R)} d^2x e^{i\mathbf{k}\cdot\mathbf{x}}(\mathbf{k}\cdot\mathbf{x})x^1$$

and in the second term we have

$$\begin{aligned} g_{i\nu}n^i\mathcal{K}^\nu &= n_i\mathcal{K}^i \\ &\sim 2x^1(x_ix^i) + x^1(\eta^2 - R^2) \\ &\sim (\eta^2 + R^2)x^1 \\ &\implies \int_{\partial B(R)} d^2x e^{i\mathbf{k}\cdot\mathbf{x}}(\eta^2 - R^2)x^1 \end{aligned}$$

In both cases the spacial integral averages to zero when taken over a large enough sphere and so there is no flux through the boundary due to the cross terms either.